**Journal of Statistics: Advances in Theory and Applications**
Volume 13, Number 2, 2015, Pages 101-133

# A NEW KUMARASWAMY TRANSMUTED MODIFIED WEIBULL DISTRIBUTION: WITH APPLICATION

**MAHMOUD M. MANSOUR[1], MOHAMED S. HAMED[1] and SALAH M. MOHAMED[2]**

[1]Department of Statistics
Mathematics and Insurance
Benha University
Egypt
e-mail: mahmoud.mansour@fcom.bu.edu.eg
moswilem@gmail.com

[2]Institute of Statistical Studies & Research
Cairo University
Egypt
e-mail: dr.mahdym62@gmail.com

## Abstract

This paper introduces a new generalization of the Kumaraswamy-transmuted exponentiated modified Weibull distribution by Al-Babtain et al. [2], based on a new family of life time distribution by Mansour et al. [29]. We refer to the new distribution as Kumaraswamy new transmuted modified Weibull (Kw-NTMW) distribution. The new model contains more than fifty lifetime distributions as special cases such as the Kumaraswamy-transmuted exponentiated modified Weibull (Kw-TEMW), transmuted modified Weibull, exponentiated modified Weibull, exponentiated Weibull, exponentiated exponential, transmuted Weibull, Rayleigh, linear failure rate, and exponential distributions, among others. The properties of the new model are discussed and the maximum likelihood estimation is used to evaluate the parameters. Explicit expressions

2010 Mathematics Subject Classification: 60-XX, 60EXX.

Keywords and phrases: Kumaraswamy distribution, transmutation, survival function, exponentiated exponential, order statistics, maximum likelihood estimation.

Received April 8, 2015

are derived for the moments and examine the order statistics. The usefulness of the Kumaraswamy-transmuted exponentiated modified Weibull distribution for modelling reliability data is illustrated by using real data.

## 1. Introduction

In many applied sciences such as medicine, engineering and finance, amongst others, modelling and analyzing lifetime data are crucial. Several lifetime distributions have been used to model such kinds of data. The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of it, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

For complex electronic and mechanical systems, the failure rate often exhibits nonmonotonic (bathtub or upside-down bathtub unimodal) failure rates (Xie and Lai [42]). Distributions with such failure rates have attracted a considerable attention of researchers in reliability engineering. In software reliability, bathtub shaped failure rate is encountered in firmware, and in embedded software in hardware devices. Firmware plays an important role in functioning of hard drives of large computers, spacecraft and high performance aircraft control systems, advanced weapon systems, safety critical control systems used for monitoring the industrial process in chemical and nuclear plants (Zhang et al. [44]). The upside down bathtub shaped failure rate is used in data of motor bus failures (Mudholkar et al. [34]), for optimal burn-in decisions (Block and Savits [7]), for ageing properties in reliability (Gupta et al. [18], Jiang et al. [21]) and the course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

The Weibull distribution is a widely used statistical model for studying fatigue and endurance life in engineering devices and materials. Many examples can be found among the electronics, materials, and automotive industries. Recent advances in Weibull theory have also created numerous specialized Weibull applications. Modern computing technology has made many of these techniques accessible across the engineering spectrum. Recently, Ammar and Mazen [3] proposed the modified Weibull distribution with cumulative distribution function (cdf) given by

$$F(x) = 1 - e^{-(\theta x^{\nu} + \gamma x)}. \tag{1}$$

Al-Babtain et al. [2], introduced the Kumaraswamy-transmuted exponentiated modified Weibull (Kw-TEMW) distribution with cumulative distribution function (cdf) and probability density function (pdf) (for $x > 0$) given by

$$F(x) = 1 - \left\{1 - [1 - e^{-(\theta x^{\nu} + \gamma x)}]^{\alpha a}\left[1 + \lambda - \lambda(1 - e^{-(\theta x^{\nu} + \gamma x)})^{\alpha}\right]^{a}\right\}^{b}, \tag{2}$$

and

$$f(x) = ab\alpha(\theta \nu x^{\nu-1} + \gamma)e^{-(\theta x^{\nu} + \gamma x)}\left\{1 + \lambda - 2\lambda(1 - e^{-(\theta x^{\nu} + \gamma x)})^{\alpha}\right\}$$

$$\times [1 - e^{-(\theta x^{\nu} + \gamma x)}]^{\alpha a - 1}\left\{1 + \lambda - \lambda(1 - e^{-(\theta x^{\nu} + \gamma x)})^{\alpha}\right\}^{a-1}$$

$$\times \left\{1 - (1 - e^{-(\theta x^{\nu} + \gamma x)})^{\alpha a}\left[1 + \lambda - \lambda(1 - e^{-(\theta x^{\nu} + \gamma x)})^{\alpha}\right]^{a}\right\}^{b-1}, \tag{3}$$

where $a, b, \alpha > 0$ and $\nu > 0$ are shape parameters, and $\theta > 0$ and $\gamma > 0$ are scale parameters and $|\lambda| \leq 1$ is a transmuted parameter.

In this article, we use transmutation map approach suggested by Mansour et al. [29] to define a new model, which generalizes the Kumaraswamy-transmuted exponentiated modified Weibull (Kw-TEMW) distribution. According to this approach, the cumulative distribution function (cdf) satisfy the relationship:

$$F(x) = (1 + \lambda)[G(x)]^{\delta} - \lambda[G(x)]^{\alpha}, \quad x > 0, \tag{4}$$

where $\alpha, \delta > 0$ for $0 > \lambda > -1$, and $\alpha > 0, (\alpha + \alpha / 4) \geq \delta \geq (\alpha / 2)$ for $0 < \lambda < 1$, $G(x)$ be the cumulative distribution function (cdf) of a non-negative absolutely continuous random variable, $G(x)$ be strictly increasing on its support, and $G(0) = 0$.

Kumaraswamy [25] introduced a two-parameter distribution on $(0, 1)$, which will be referred to by "Kw" in the sequel. Its cdf is given by

$$F(x) = 1 - (1 - x^{a})^{b}, \quad x \in (0, 1), \tag{5}$$

where $a > 0$ and $b > 0$ are shape parameters. The model in (5) compares extremely favorably in terms of simplicity with the beta cdf, that is, the incomplete beta function ratio. The pdf corresponding to (5) is given by

$$f(x) = ab(1 - x^{a})^{b-1}, \quad x \in (0, 1). \tag{6}$$

The Kw density function has the same basic shape properties of the beta distribution: $a > 1$ and $b > 1$ (unimodal); $a < 1$ and $b < 1$ (uniantimodal); $a > 1$ and $b \leq 1$ (increasing); $a \leq 1$ and $b > 1$ (decreasing); $a = b = 1$ (constant). The Kw distribution does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before. However, Jones [22] explored the background and genesis of the Kw distribution and, more importantly, highlighted some advantages and disadvantages of the beta and Kw distributions.

For an arbitrary baseline cdf, $G(x)$, Cordeiro and Castro [10] defined the Kw-$G$ distribution by the pdf $f(x)$ and cdf $F(x)$ as

$$f(x) = ab.g(x).G^{a-1}(x)[1 - G^{a}(x)]^{b-1}, \tag{7}$$

and

$$F(x) = 1 - [1 - G^{a}(x)]^{b}, \tag{8}$$

respectively, where $g(x) = dG(x) / dx$ and $a$ and $b$ are two extra positive shape parameters. It follows immediately from (8) that the Kw-$G$ distribution with parent cdf $G(x) = x$ produces the minimax distribution (6). If $X$ is a random variable with pdf (7), we write $X \sim Kw - G(a, b)$, where $a$ and $b$ are additional shape parameters which aim to govern skewness and tail weight of the generated distribution.

The rest of the article is organized as follows. In Section 2, introduces the proposed Kumaraswamy new transmuted modified Weibull model according to the new class of distribution and Kumaraswamy distribution. In Section 3, we find the reliability function, hazard rate and cumulative hazard rate of the subject model. The expansion for the pdf and the cdf functions is derived in Section 4. In Section 5, the statistical properties include quantile functions, median, moments, and moment generating function are given. In Section 6, order statistics are discussed. In Section 7, we introduce the method of likelihood estimation as point estimation and, given the equation used to estimate the parameters, using the maximum product spacing estimates and the least square estimates techniques. In Section 8, we fit the distribution to real data set to examine it and to suitability it with some models. Finally, we conclude the paper.

## 2. A Kumaraswamy New Transmuted Modified Weibull Distribution

In this section, we introduce a new distribution called the Kumaraswamy new transmuted modified Weibull distribution denoted by (Kw-NTMW) distribution as a generalization of the Kw-TEMW distribution. By using (1), (4), and (8), then the cumulative distribution function of Kw-NTMW model (for $x > 0$) denoted by $F(x, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) \equiv F(x)$ becomes

$$F(x) = 1 - \left\{1 - \left\{(1+\lambda)\left[I(x,\ \vartheta)\right]^{\delta} - \lambda\left[I(x,\ \vartheta)\right]^{\alpha}\right\}^{a}\right\}^{b}, \tag{9}$$

where

$$I(x,\ \vartheta) = 1 - e^{-(\theta x^{\nu} + \gamma x)},$$

whereas its pdf can be expressed

$$f(x) = a.b.(\theta\nu x^{\nu-1} + \gamma)e^{-(\theta x^{\nu}+\gamma x)}[(1+\lambda)\delta[I(x,\ \vartheta)]^{\delta-1} - \lambda\alpha[I(x,\ \vartheta)]^{\alpha-1}]$$

$$\times \left\{(1+\lambda)\left[I(x,\ \vartheta)\right]^{\delta} - \lambda\left[I(x,\ \vartheta)\right]^{\alpha}\right\}^{a-1}$$

$$\times \left\{1 - \left\{(1+\lambda)\left[I(x,\ \vartheta)\right]^{\delta} - \lambda\left[I(x,\ \vartheta)\right]^{\alpha}\right\}^{a}\right\}^{b-1}, \tag{10}$$

where $\nu$, $\delta > 0$, $\alpha > 0$, $a > 0$ and $b > 0$ are shape parameters and $\theta > 0$ and $\gamma > 0$ are scale parameters and $|\lambda|$ is a transmuted parameter. The random variable $x$ with the density function (10) is said to have a Kumaraswamy new transmuted modified Weibull Kw-NTMW distribution.

The importance of the proposed Kw-NTMW model that it is very flexible model that approaches to different distributions when its parameters are changed. The flexibility of the Kw-NTMW is explained in Table 1 when their parameters are carefully chosen.

**Table 1.** The special cases of the Kw-NTMW distribution

| **Distribution** | **Parameters** | | | | | | | | **Author** |
|---|---|---|---|---|---|---|---|---|---|
| | $a$ | $b$ | $\lambda$ | $\delta$ | $\alpha$ | $\theta$ | $\gamma$ | $\nu$ | |
| **Kw-TEMW** | | | | $\alpha/2$ | | | | | Al-Babtain et al. [2] |
| **Kw-TELFR** | | | | 1 | | | | 2 | |
| **Kw-TEW** | | | | 1 | | 0 | | | New |
| **Kw-TER** | | | | 1 | | 0 | | 2 | New |
| **Kw-TEE** | | | | 1 | | | 0 | | New |
| **Kw-TMW** | | | | 1 | 2 | | | | New |
| **Kw-TLFR** | | | | 1 | 2 | | | 2 | |
| **Kw-TW** | | | | 1 | 2 | 0 | | | |
| **Kw-TR** | | | | 1 | 2 | 0 | | 2 | |
| **Kw-TE** | | | | 1 | 2 | | 0 | | |
| **Kw-EMW** | | | 0 | $\alpha/2$ | | | | | |
| **Kw-ELFR** | | | 0 | | | | | 2 | Elbatal [15] |
| **Kw-EW** | | | 0 | $\alpha/2$ | | 0 | | | New |
| **Kw-ER** | | | 0 | $\alpha/2$ | | 0 | | 2 | Gomes et al. [16] |
| **Kw-EE** | | | 0 | | | | 0 | | |
| **Kw-MW** | | | 0 | 1 | | | | | New |
| **Kw-LFR** | | | 0 | 1 | | | | 2 | |
| **Kw-MW** | | | 0 | 1 | | 0 | | | Cordeiro et al. [13] |
| **Kw-R** | | | 0 | 1 | | 0 | | 2 | |
| **Kw-E** | | | 0 | 1 | | | 0 | | |
| **ETEMW** | | 1 | | $\alpha/2$ | | | | | New |
| **ETELFR** | | 1 | | $\alpha/2$ | | | | 2 | New |
| **ETEW** | | 1 | | $\alpha/2$ | | 0 | | | New |
| **ETER** | | 1 | | $\alpha/2$ | | 0 | | 2 | New |
| **ETEE** | | 1 | | $\alpha/2$ | | | 0 | | New |
| **ETMW** | | 1 | | 1 | 2 | | | | New |
| **ETLFR** | | 1 | | 1 | 2 | | | 2 | New |
| **ETW** | | 1 | | 1 | 2 | 0 | | | |
| **ETR** | | 1 | | 1 | 2 | 0 | | 2 | |

**Table 1.** (Continued)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **ETE** | | 1 | | 1 | 2 | | 0 | | |
| **EMW** | | 1 | 0 | | | | | | |
| **ELFR** | | 1 | 0 | | | | | 2 | |
| **EW** | | 1 | 0 | | | 0 | | | |
| **ER** | | 1 | 0 | | | 0 | | 2 | |
| **EE** | | 1 | 0 | | | | 0 | | |
| **EMW** | | 1 | 0 | 1 | | | | | Elbatal [14] |
| **EW** | | 1 | 0 | 1 | | 0 | | | Mudholkar and Srivastava [33] |
| **EE** | | 1 | 0 | 1 | | | 0 | | Gupta and Kundu [19] |
| **TEMW** | 1 | 1 | | $\alpha/2$ | | | | | Ashour and Eltehiwy [5] |
| **TELFR** | 1 | 1 | | $\alpha/2$ | | | | 2 | |
| **TEW** | 1 | 1 | | $\alpha/2$ | | 0 | | | |
| **TER** | 1 | 1 | | $\alpha/2$ | | 0 | | 2 | Merovci [32] |
| **TEE** | 1 | 1 | | $\alpha/2$ | | | 0 | | Merovci [30] |
| **TMW** | 1 | 1 | | 1 | 2 | | | | Khan and King [24] |
| **TLFR** | 1 | 1 | | 1 | 2 | | | 2 | |
| **TW** | 1 | 1 | | 1 | 2 | 0 | | | Aryal and Tsokos [4] |
| **TR** | 1 | 1 | | 1 | 2 | 0 | | 2 | |
| **TE** | 1 | 1 | | 1 | 2 | | 0 | | |
| **ELFR** | 1 | 1 | 0 | | | | | 2 | Sarhan and Kundu [37] |
| **ER** | 1 | 1 | 0 | | | 0 | | 2 | Kundu and Raqab [26] |
| **MW** | 1 | 1 | 0 | 1 | | | | | Sarhan and Zaindin [38] |
| **LFR** | 1 | 1 | 0 | | 1 | | | 2 | |
| **W** | 1 | 1 | 0 | 1 | | 0 | | | Weibull [41] |
| **R** | 1 | 1 | 0 | | 1 | 0 | | 2 | |
| **E** | 1 | 1 | 0 | 1 | | | 0 | | |

Figures 1 and 2 illustrates some of the possible shapes of the pdf and cdf of the Kw-NTMW distribution for selected values of the parameters $\lambda$, $\theta$, $\nu$, $\gamma$, $\delta$, $\alpha$, $a$, and $b$, respectively.

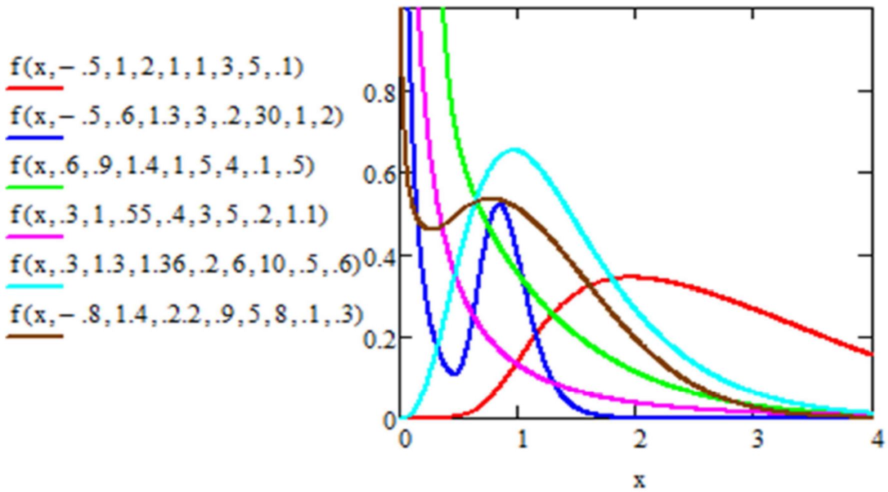


**Figure 1.** Probability density function of the Kw-NTMW distribution.

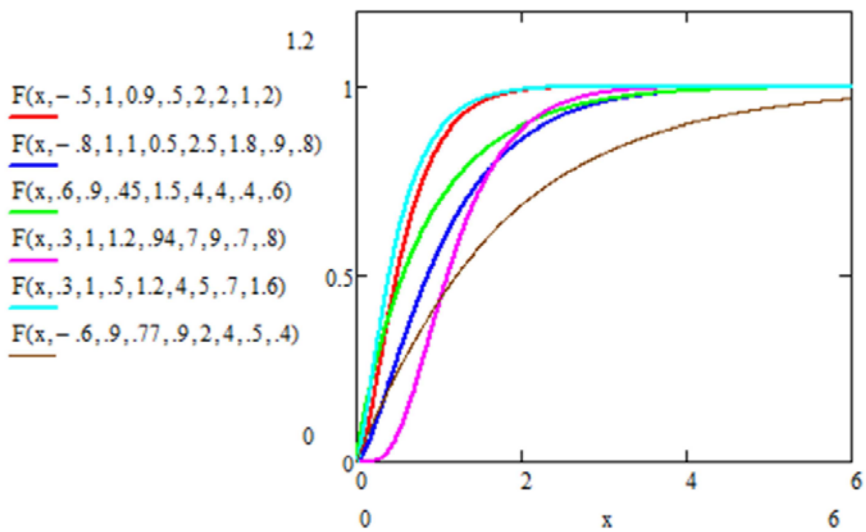


**Figure 2.** Distribution function of the Kw-NTMW distribution.

## 3. Reliability Analysis

The characteristics in reliability analysis which are the reliability function (RF), the hazard rate function (HF), and the cumulative hazard rate function (CHF) for the Kw-NTMW are introduces in this section.

### 3.1. Reliability function

The reliability function (RF) also known as the survival function, which is the probability of an item not failing prior to some time $t$, is defined by $R(x) = 1 - F(x)$. The reliability function of the Kw-NTMW distribution denoted by $R_{\text{Kw-NTMW}}(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$, can be a useful characterization of life time data analysis. It can be defined as

$$R_{\text{Kw-NTMW}}(x, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) = 1 - F_{\text{Kw-NTMW}}(x, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b),$$

the survival function of is given by

$$R_{\text{Kw-NTMW}}(x, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) = \left\{1 - \left\{(1+\lambda)[I(x, \vartheta)]^{\delta} - \lambda[I(x, \vartheta)]^{\alpha}\right\}^{a}\right\}^{b}.$$

Figure 3 illustrates the pattern of the called the Kumaraswamy new transmuted modified Weibull distribution (Kw-NTMW) distribution reliability function with different choices of parameters $\lambda, \theta, \nu, \gamma, \delta, \alpha, a$, and $b$.

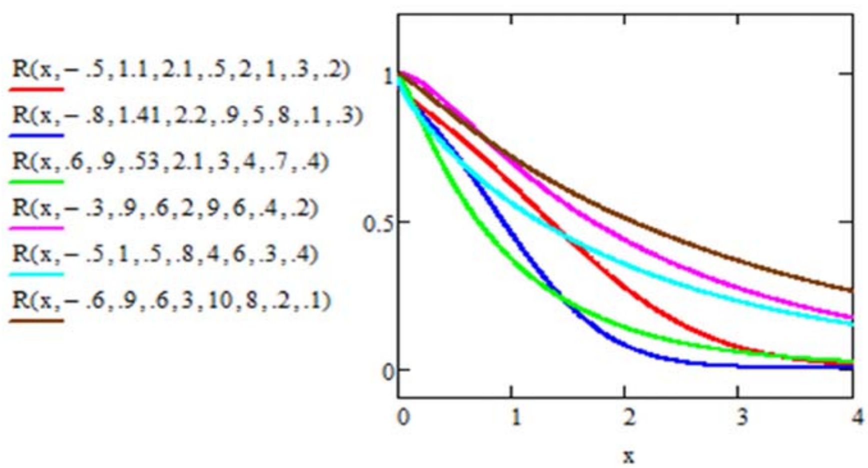


**Figure 3.** Reliability function of the Kw-NTMW distribution.

### 3.2. Hazard rate function

Suppose a system is made up of $b$ independent sub-systems functioning independently at a given time and that each sub-system is made up of $a$ independent parallel components. If we want to improve the reliable of the given system we had to duplicate each component in parallel form, then the time to failure of the given system will have the cumulative distribution function (10) at $\lambda = -1$. The failure rate function associated with (10) is given by

$$h(x) = a.b.(\theta \nu x^{\nu-1} + \gamma) e^{-(\theta x^{\nu} + \gamma x)} [(1+\lambda)\delta[I(x, \vartheta)]^{\delta-1} - \lambda\alpha[I(x, \vartheta)]^{\alpha-1}]$$

$$\times \frac{\{(1+\lambda)[I(x, \vartheta)]^{\delta} - \lambda[I(x, \vartheta)]^{\alpha}\}^{a-1}}{1 - \{(1+\lambda)[I(x, \vartheta)]^{\delta} - \lambda[I(x, \vartheta)]^{\alpha}\}^{a}}.$$

Figure 4 illustrates some of the possible shapes of the hazard rate function of the Kumaraswamy new transmuted modified Weibull distribution for different values of the parameters $\lambda$, $\theta$, $\nu$, $\gamma$, $\delta$, $\alpha$, $a$, and $b$.

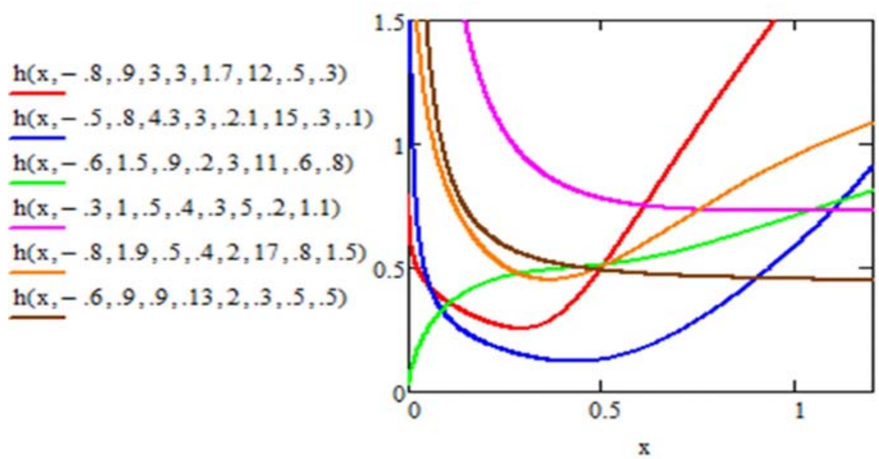


**Figure 4.** Hazard rate of the Kw-NTMW distribution.

Another importance of The Kw-NTMW model due to its flexibility in accommodating all forms of the hazard rate function as seen from Figure 4 (by changing its parameter values) seems to be an important distribution that can be used.

### 3.3. Cumulative hazard rate function

The cumulative hazard function (CHF) of the Kumaraswamy new transmuted modified Weibull distribution, denoted by $H_{\text{Kw-NTMW}}(x, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$, is defined as

$$H_{\text{Kw-NTMW}}(x, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) = \int_0^x h_{\text{Kw-NTMW}}(x, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)dx$$

$$= -\ln R_{\text{Kw-NTMW}}(x, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b),$$

$$H_{\text{Kw-NTMW}}(x, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) = -\ln\left[\left\{1 - \left\{(1+\lambda)[I(x, \vartheta)]^{\delta} - \lambda[I(x, \vartheta)]^{\alpha}\right\}^{a}\right\}^{b}\right].$$

## 4. Expansion for the pdf and the cdf Functions

In this section, we introduced another expression for the pdf and the cdf functions using. The Maclaurin expansion and binomial expansion to simplifying the pdf and the cdf forms.

### 4.1. Expansion for the pdf function

From Equation (10) and using the expansion

$$(1-z)^k = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(k+1)}{\Gamma(k-j+1)j!} z^j, \tag{11}$$

which holds for $|z| < 1$ and $k > 0$. Using (11) in Equation (10), then the pdf function of the Kw-NTMW can be written as

$$f(x) = a.b.(\theta\nu x^{\nu-1} + \gamma)e^{-(\theta x^{\nu} + \gamma x)}[(1+\lambda)\delta[I(x, \vartheta)]^{\delta-1} - \lambda\alpha[I(x, \vartheta)]^{\alpha-1}]$$

$$\times \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{i!\, \Gamma(b-i)} \left\{(1+\lambda)[I(x, \vartheta)]^{\delta} - \lambda[I(x, \vartheta)]^{\alpha}\right\}^{\alpha i + a - i}. \tag{12}$$

Equation (12) can be written as

$$f(x) = a.b.(\theta \nu x^{\nu-1} + \gamma) e^{-(\theta x^{\nu} + \gamma x)} [(1+\lambda)\delta [I(x, \vartheta)]^{\delta-1} - \lambda\alpha [I(x, \vartheta)]^{\alpha-1}]$$

$$\times \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{i!\, \Gamma(b-i)} (1+\lambda)^{\alpha(i+1)-1} [I(x, \vartheta)]^{\delta(\alpha(i+1)-1)}$$

$$\times \left\{ 1 - \frac{\lambda [I(x, \vartheta)]^{\alpha}}{(1+\lambda)[I(x, \vartheta)]^{\delta}} \right\}^{\alpha(i+1)-1}, \tag{13}$$

which holds for $\frac{\lambda [I(x, \vartheta)]^{\alpha}}{(1+\lambda)[I(x, \vartheta)]^{\delta}} < 1.$

Using (11) and applying it to (13), the pdf of the Kw-NTMW model can be written as

$$f(x) = a.b.(\theta \nu x^{\nu-1} + \gamma) e^{-(\theta x^{\nu} + \gamma x)} [(1+\lambda)\delta [I(x, \vartheta)]^{\delta-1} - \lambda\alpha [I(x, \vartheta)]^{\alpha-1}]$$

$$\times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} \Gamma(b) \Gamma(a(i+1))}{i!\, j!\, \Gamma(b-i) \Gamma(a(i+1)-j)}$$

$$\times \lambda^{j} (1+\lambda)^{\alpha(i+1)-j-1} [I(x, \vartheta)]^{\delta(a(i+1)-1)+\alpha j - \delta j}, \tag{14}$$

using binomial expansion and applying it to (14), the pdf of the Kw-NTMW model can be written as

$$f(x) = a.b.(\theta \nu x^{\nu-1} + \gamma) e^{-(\theta x^{\nu} + \gamma x)}$$

$$\times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{1} \frac{(-1)^{i+j+k} \Gamma(b) \Gamma(a(i+1))}{i!\, j!\, \Gamma(b-i) \Gamma(a(i+1)-j)}$$

$$\times \lambda^{j+k} (1+\lambda)^{\alpha(i+1)-j-k}$$

$$\times [I(x, \vartheta)]^{\delta(a(i+1)-1)+\alpha j - \delta j + (\alpha-1)k + (\delta-1)(1-k)}, \tag{15}$$

using (11) and applying it to (15), the pdf of the Kw-NTMW model can be written as

$$f(x) = a.b.(\theta\nu x^{\nu-1} + \gamma)\sum_{i,j,l=0}^{\infty}\sum_{k=0}^{1}\frac{(-1)^{i+j+k+l}\Gamma(b)\Gamma(a(i+1))}{i!\,l!\,j!\,\Gamma(b-i)\Gamma(a(i+1)-j)}$$

$$\times\frac{\Gamma(\delta(a(i+1)-1)+\alpha j-\delta j+(\alpha-1)k+(\delta-1)(1-k)-1)}{\Gamma(\delta(a(i+1)-1)+\alpha j-\delta j+(\alpha-1)k+(\delta-1)(1-k)-l-1)}$$

$$\times\lambda^{j+k}(1+\lambda)^{a(i+1)-j-k}e^{-(\theta x^{\nu}+\gamma x)(l+1)}, \tag{16}$$

the pdf of Kw-NTMW distribution can then be represented as

$$f(x) = \sum_{i,j,l=0}^{\infty}\sum_{k=0}^{1}A_{i,k}(\theta\nu \mathrm{x}^{\nu-1} + \gamma)e^{-(\theta x^{\nu}+\gamma x)(l+1)}, \tag{17}$$

where $A_{i,k}$ is a constant term given by

$$A_{i,k} = a.b.\frac{(-1)^{i+j+k+l}\Gamma(b)\Gamma(a(i+1))}{i!\,l!\,j!\,\Gamma(b-i)\Gamma(a(i+1)-j)}\lambda^{j+k}(1+\lambda)^{a(i+1)-j-k}$$

$$\times\frac{\Gamma(\delta(a(i+1)-1)+\alpha j-\delta j+(\alpha-1)k+(\delta-1)(1-k)-1)}{\Gamma(\delta(a(i+1)-1)+\alpha j-\delta j+(\alpha-1)k+(\delta-1)(1-k)-l-1)'},$$

for $\dfrac{\lambda[I(x,\vartheta)]^{\alpha}}{(1+\lambda)[I(x,\vartheta)]^{\delta}} < 1;$

$$A_{i,k} = a.b.\frac{(-1)^{i+j+k+l}\Gamma(b)\Gamma(a(i+1))}{i!\,l!\,j!\,\Gamma(b-i)\Gamma(a(i+1)-j)}(-\lambda)^{a(i+1)-j-k+1}(1+\lambda)^{j-k+1}$$

$$\times\frac{\Gamma(\alpha(a(i+1)-1)-\alpha j+\delta j+(\alpha-1)k+(\delta-1)(1-k)-1)}{\Gamma(\alpha(a(i+1)-1)-\alpha j+\delta j+(\alpha-1)k+(\delta-1)(1-k)-l-1)'},$$

for $\dfrac{\lambda[I(x,\vartheta)]^{\alpha}}{(1+\lambda)[I(x,\vartheta)]^{\delta}} > 1.$

**4.2. Expansion for the cdf function**

Using expansion (11) to Equation (9), then the cdf function of the new transmuted modified Weibull distribution can be written as

$$F(x) = 1 - \sum_{i, j, k, l=0}^{\infty} B_{i:l}(\theta x^{\nu} + \gamma x)^{l}, \tag{18}$$

where $B_{i:l}$ is a constant term given by

$$B_{i:l} = \frac{(-1)^{i+j+k+l}\Gamma(b+1)\Gamma(a(i+1)+1)\Gamma(\delta(a(i+1)) - \delta j + \alpha j + 1)}{i!\, j!\, k!\, l!\, \Gamma(b-i+1)\Gamma(a(i+1) - j + 1)\Gamma(\delta(a(i+1)) - \delta j + \alpha j - k + 1)}$$

$$\times k^{l}\lambda^{j}(1+\lambda)^{a(i+1)-j},$$

for $\dfrac{\lambda[I(x, \vartheta)]^{\alpha}}{(1+\lambda)[I(x, \vartheta)]^{\delta}} < 1;$

$$B_{i:l} = \frac{(-1)^{i+j+k+l}\Gamma(b+1)\Gamma(a(i+1)+1)\Gamma(\alpha(a(i+1)) - \alpha j + \delta j + 1)}{i!\, j!\, k!\, l!\, \Gamma(b-i+1)\Gamma(a(i+1) - j + 1)\Gamma(\alpha(a(i+1)) - \alpha j + \delta j - k + 1)}$$

$$\times k^{l}(-\lambda)^{a(i+1)-j}(1+\lambda)^{j},$$

for $\dfrac{\lambda[I(x, \vartheta)]^{\alpha}}{(1+\lambda)[I(x, \vartheta)]^{\delta}} > 1.$

## 5. Statistical Properties

In this section, we discuss the most important statistical properties of the Kw-NTMW distribution.

### 5.1. Quantile function

The quantile function is obtained by inverting the cumulative distribution (18), where the $p$-th quantile $x_p$ of Kw-NTMW model is the real solution of the following equation:

$$1 - \sum_{i, j, k, l=0}^{\infty} B_{i:l}(\theta x_p{}^{\nu} + \gamma x_p)^{l} - p = 0. \tag{19}$$

An expansion for the median $M$ follows by taking $p = 0.5$.

**5.2. Moments**

Using (17), the $r$-th non-central moments $\mu'_r = E(X^r)$ of the Kw-NTMW model is given by

$$\mu'_r = \sum_{i,j,l,m=0}^{\infty} \sum_{k=0}^{1} A_{i,k} \frac{(-1)^m (l+1)^m \theta^m}{m!} \left[ \frac{\gamma\Gamma(r + m\nu + 1)}{(\gamma(l+1))^{r+m\nu+1}} + \frac{\theta\nu\Gamma(\nu + m\nu + r)}{(\gamma(l+1))^{\nu+m\nu+r}} \right]. \tag{20}$$

In particular, when $r = 1$, Equation (20) yields the mean of the Kw-NTMW distribution $\mu$ as

$$\mu = \sum_{i,j,l,m=0}^{\infty} \sum_{k=0}^{1} A_{i,k} \frac{(-1)^m (l+1)^m \theta^m}{m!} \left[ \frac{\gamma\Gamma(m\nu + 2)}{(\gamma(l+1))^{m\nu+2}} + \frac{\theta\nu\Gamma(\nu + m\nu + 1)}{(\gamma(l+1))^{\nu+m\nu+1}} \right].$$

The $n$-th central moments or (moments about the mean) can be obtained easily from the $r$-th non-central moments throw the relation

$$m_u = E(X - \mu)^n = \sum_{r=0}^{n} (-\mu)^{n-r} E(X^r).$$

Then the $n$-th central moments of the Kw-NTMW is given by

$$m_u = \sum_{r=0}^{n} (-\mu)^{n-r} \sum_{i,j,l,m=0}^{\infty} \sum_{k=0}^{1} A_{i,k} \frac{(-1)^m (l+1)^m \theta^m}{m!} \left[ \frac{\gamma\Gamma(m\nu + 2)}{(\gamma(l+1))^{m\nu+2}} + \frac{\theta\nu\Gamma(\nu + m\nu + 1)}{(\gamma(l+1))^{\nu+m\nu+1}} \right].$$

**5.3. The moment generating function**

If $X$ is from a Kw-NTMW distribution, then its mgf is

$$M_x(t) = \sum_{i,j,l,m,r=0}^{\infty} \sum_{k=0}^{1} A_{i,k} \frac{(-1)^m (l+1)^m \theta^m t^r}{m!\, r!} \left[ \frac{\gamma\Gamma(r + m\nu + 1)}{(\gamma(l+1))^{r+m\nu+1}} + \frac{\theta\nu\Gamma(\nu + m\nu + r)}{(\gamma(l+1))^{\nu+m\nu+r}} \right].$$

## 6. Order Statistics

The order statistics and their moments have great importance in many statistical problems and they have many applications in reliability analysis and life testing. The order statistics arise in the study of reliability of a system. The order statistics can represent the lifetimes of units or components of a reliability system. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size $n$ from the Kw-NTMW $(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$ with cumulative distribution function (cdf), and the corresponding probability density function (pdf), as in (9) and (10), respectively. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the corresponding order statistics. Then the pdf of $Y_{(r:n)}, 1 \leq r \leq n,$ denoted by $f_{r:n}(y)$ is given by

$$f_{r:n}(y) = C_{r:n} f_{\mathrm{Kw-NTMW}}(\lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)[F_{\mathrm{Kw-NTMW}}(\lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)]^{r-1}$$

$$\times [R_{\mathrm{Kw-NTMW}}(\lambda, \theta, \nu, \gamma, \beta, \delta, \alpha)]^{n-r}.$$

Therefore, the pdf of the largest order statistic $X_n$ is given by

$$f_{X_n(x)=} n.a.b.(\theta\nu \mathrm{x}^{\nu-1} + \gamma) e^{-(\theta x^{\nu} + \gamma x)} \left[(1+\lambda)\delta[I(x, \vartheta)]^{\delta-1} - \lambda\alpha[I(x, \varphi)]^{\alpha-1}\right]$$

$$\times \left\{(1+\lambda)[I(x, \vartheta)]^{\delta} - \lambda[I(x, \vartheta)]^{\alpha}\right\}^{a-1}$$

$$\times \left\{1 - \left\{(1+\lambda)[I(x, \vartheta)]^{\delta} - \lambda[I(x, \varphi)]^{\alpha}\right\}^{a}\right\}^{b-1}$$

$$\times \left[1 - \left\{1 - \left\{(1+\lambda)[I(x, \vartheta)]^{\delta} - \lambda[I(x, \varphi)]^{\alpha}\right\}^{a}\right\}^{b}\right]^{n-1}. \qquad (21)$$

While, the pdf of the smallest order statistic $X_1$ is given by

$$f_{X_1(x)=} n.a.b.(\theta\nu \mathrm{x}^{\nu-1} + \gamma) e^{-(\theta x^{\nu} + \gamma x)} \left[(1+\lambda)\delta[I(x, \vartheta)]^{\delta-1} - \lambda\alpha[I(x, \varphi)]^{\alpha-1}\right]$$

$$\times \left\{(1+\lambda)[I(x, \vartheta)]^{\delta} - \lambda[I(x, \vartheta)]^{\alpha}\right\}^{a-1}$$

$$\times\left\{1-\left\{(1+\lambda)\left[I(x,\ \vartheta)\right]^{\delta}-\lambda\left[I(x,\ \varphi)\right]^{\alpha}\right\}^{a}\right\}^{b-1}$$

$$\times\left[\left\{1-\left\{(1+\lambda)\left[I(x,\ \vartheta)\right]^{\delta}-\lambda\left[I(x,\ \varphi)\right]^{\alpha}\right\}^{a}\right\}^{b}\right]^{n-1}. \tag{22}$$

## 7. Estimation of the Parameters

In this section, we introduce the method of likelihood to estimate the parameters involved, then give the equations used to estimate the parameters by using the maximum product spacing estimates and the least square estimates techniques.

### 7.1. Maximum likelihood estimation

The maximum likelihood estimators (MLEs) for the parameters of the Kumaraswamy new transmuted modified Weibull distribution $\text{Kw-NTMW}(\lambda,\ \theta,\ \nu,\ \gamma,\ \delta,\ \alpha,\ a,\ b)$ is discussed in this section. Consider the random sample $x_1, x_2, \ldots, x_n$ of size $n$ from new transmuted exponentiated modified distribution $\text{Kw-NTMW}(\lambda,\ \theta,\ \nu,\ \gamma,\ \delta,\ \alpha,\ a,\ b)$ with probability density function in (10), then the likelihood function can be expressed as follows:

$$L(x_1,\ x_2,\ \ldots,\ x_n,\ \lambda,\ \theta,\ \nu,\ \gamma,\ \delta,\ \alpha,\ a,\ b)=\prod_{i=1}^{n} f_{\text{Kw-NTMW}}(x_i,\ \lambda,\ \theta,\ \nu,\ \gamma,\ \delta,\ \alpha,\ a,\ b),$$

$$L(x_1,\ x_2,\ \ldots,\ x_n,\ \lambda,\ \theta,\ \nu,\ \gamma,\ \delta,\ \alpha,\ a,\ b)=\prod_{i=1}^{n} a.b.(\theta\nu x_i{}^{\nu-1}+\gamma)e^{-(\theta x_i{}^{\nu}+\gamma x_i)}$$

$$\times\left[(1+\lambda)\delta\left[I(x_i,\ \vartheta)\right]^{\delta-1}-\lambda\alpha\left[I(x_i,\ \vartheta)\right]^{\alpha-1}\right]$$

$$\times\left\{(1+\lambda)\left[I(x_i,\ \vartheta)\right]^{\delta}-\lambda\left[I(x_i,\ \vartheta)\right]^{\alpha}\right\}^{a-1}$$

$$\times\left\{1-\left\{(1+\lambda)\left[I(x_i,\ \vartheta)\right]^{\delta}-\lambda\left[I(x_i,\ \vartheta)\right]^{\alpha}\right\}^{a}\right\}^{b-1},$$

where $I(x_i,\ \vartheta)=1-e^{-(\theta x_i{}^{\nu}+\gamma x_i)}$.

Hence, the log-likelihood function $\tau = \ln L$ becomes

$$\tau = nlnb + nlna + \sum_{i=1}^{n} \ln(\theta \nu x_i^{\nu-1} + \gamma) - \sum_{i=1}^{n} (\theta x_i^{\nu} + \gamma x_i)$$

$$+ \sum_{i=1}^{n} \ln\left[(1+\lambda)\delta[I(x_i, \vartheta)]^{\delta-1} - \lambda\alpha[I(x_i, \vartheta)]^{\alpha-1}\right]$$

$$+ (a-1)\sum_{i=1}^{n} \ln\left\{(1+\lambda)[I(x_i, \vartheta)]^{\delta} - \lambda[I(x_i, \vartheta)]^{\alpha}\right\}$$

$$+ (b-1)\sum_{i=1}^{n} \ln\left\{1 - \left\{(1+\lambda)[I(x_i, \vartheta)]^{\delta} - \lambda[I(x_i, \vartheta)]^{\alpha}\right\}^{a}\right\}. \tag{23}$$

Differentiating Equation (23) with respect to $\lambda$, $\theta$, $\nu$, $\gamma$, $\delta$, $\alpha$, $a$, and $b$, then equating it to zero, we obtain the MLEs of $\lambda$, $\theta$, $\nu$, $\gamma$, $\delta$, $\alpha$, $a$, and $b$ as follows:

$$\frac{\partial \tau}{\partial \lambda} = \sum_{i=1}^{n} \left[\frac{\delta[I(x_i, \vartheta)]^{\delta-1} - \alpha[I(x_i, \vartheta)]^{\alpha-1}}{[(1+\lambda)\delta[I(x_i, \vartheta)]^{\delta-1} - \lambda\alpha[I(x_i, \vartheta)]^{\alpha-1}]}\right]$$

$$+ (a-1)\sum_{i=1}^{n} \frac{[I(x_i, \vartheta)]^{\delta} - [I(x_i, \vartheta)]^{\alpha}}{(1+\lambda)[I(x_i, \vartheta)]^{\delta} - \lambda[I(x_i, \vartheta)]^{\alpha}}$$

$$- a.(b-1)\sum_{i=1}^{n} \frac{\left\{(1+\lambda)[I(x_i, \vartheta)]^{\delta} - \lambda[I(x_i, \vartheta)]^{\alpha}\right\}^{a-1}\left([I(x_i, \vartheta)]^{\delta} - [I(x_i, \vartheta)]^{\alpha}\right)}{1 - \left\{(1+\lambda)[I(x_i, \vartheta)]^{\delta} - \lambda[I(x_i, \vartheta)]^{\alpha}\right\}^{a}}, \tag{24}$$

$$\frac{\partial \tau}{\partial \theta} = \sum_{i=1}^{n} \frac{\nu x_i^{\nu-1}}{(\theta \nu x_i^{\nu-1} + \gamma)} - \sum_{i=0}^{\infty} x_i^{\nu}$$

$$+ \sum_{i=1}^{n} \left[\frac{x_i^{\nu} e^{-(\theta x_i^{\nu} + \gamma x_i)}\left[\lambda(\delta-1)\delta[I(x_i, \vartheta)]^{\delta-2} - \lambda(\alpha-1)\alpha[I(x_i, \vartheta)]^{\alpha-2}\right]}{(1+\lambda)\delta[I(x_i, \vartheta)]^{\delta-1} - \lambda\alpha[I(x_i, \vartheta)]^{\alpha-1}}\right]$$

$$+ (a-1)\sum_{i=1}^{n}\frac{x_i{}^{\nu}e^{-(\theta x_i{}^{\nu}+\gamma x_i)}\left(\delta[I(x_i,\vartheta)]^{\delta-1}-\alpha[I(x_i,\vartheta)]^{\alpha-1}\right)}{(1+\lambda)[I(x_i,\vartheta)]^{\delta}-\lambda[I(x_i,\vartheta)]^{\alpha}}$$

$$- a.(b-1)\sum_{i=1}^{\infty}$$

$$\frac{\left\{(1+\lambda)[I(x_i,\vartheta)]^{\delta}-\lambda[I(x_i,\vartheta)]^{\alpha}\right\}^{a-1}x_i{}^{\nu}e^{-(\theta x_i{}^{\nu}+\gamma x_i)}\left(\delta[I(x_i,\vartheta)]^{\delta-1}-\alpha[I(x_i,\vartheta)]^{\alpha-2}\right)}{1-\left\{(1+\lambda)[I(x_i,\vartheta)]^{\delta}-\lambda[I(x_i,\vartheta)]^{\alpha}\right\}^{a}},$$

(25)

$$\frac{\partial\tau}{\partial\nu} = \sum_{i=1}^{n}\frac{x_i{}^{\nu-1}\theta(1+\nu\ln x_i)}{(\theta\nu x_i{}^{\nu-1}+\gamma)} - \sum_{i=0}^{\infty}x_i{}^{\nu}\theta\ln x_i$$

$$+ \sum_{i=1}^{n}\frac{e^{-(\theta x_i{}^{\nu}+\gamma x_i)}(x_i{}^{\nu}\theta\ln x_i)\left[\lambda(\delta-1)\delta[I(x_i,\vartheta)]^{\delta-2}-\lambda(\alpha-1)\alpha[I(x_i,\vartheta)]^{\alpha-2}\right]}{(1+\lambda)\delta[I(x_i,\vartheta)]^{\delta-1}-\lambda\alpha[I(x_i,\vartheta)]^{\alpha-1}}$$

$$+ (a-1)\sum_{i=1}^{n}\frac{\left((x_i{}^{\nu}\theta\ln x_i)\right)e^{-(\theta x_i{}^{\nu}+\gamma x_i)}\left(\delta[I(x_i,\vartheta)]^{\delta-1}-\alpha[I(x_i,\vartheta)]^{\alpha-1}\right)}{(1+\lambda)[I(x_i,\vartheta)]^{\delta}-\lambda[I(x_i,\vartheta)]^{\alpha}}$$

$$- a.(b-1)\sum_{i=1}^{n}\frac{\left\{(1+\lambda)[I(x_i,\vartheta)]^{\delta}-\lambda[I(x_i,\vartheta)]^{\alpha}\right\}^{a-1}}{1-\left\{(1+\lambda)[I(x_i,\vartheta)]^{\delta}-\lambda[I(x_i,\vartheta)]^{\alpha}\right\}^{a}}$$

$$\times (x_i{}^{\nu}\theta\ln x_i)e^{-(\theta x_i{}^{\nu}+\gamma x_i)}\left(\delta[I(x_i,\vartheta)]^{\delta-1}-\alpha[I(x_i,\vartheta)]^{\alpha-1}\right), \tag{26}$$

$$\frac{\partial\tau}{\partial\gamma} = \sum_{i=1}^{n}\frac{1}{(\theta\nu x_i{}^{\nu-1}+\gamma)} - \sum_{i=0}^{n}x_i$$

$$+ \sum_{i=1}^{n}\left[\frac{x_i e^{-(\theta x_i{}^{\nu}+\gamma x_i)}\left[\lambda(\delta-1)\delta[I(x_i,\vartheta)]^{\delta-2}-\lambda(\alpha-1)\alpha[I(x_i,\vartheta)]^{\alpha-2}\right]}{\left[(1+\lambda)\delta[I(x_i,\vartheta)]^{\delta-1}-\lambda\alpha[I(x_i,\vartheta)]^{\alpha-1}\right]}\right]$$

$$+ (a-1)\sum_{i=1}^{n}\frac{(x_i)e^{-(\theta x_i{}^{\nu}+\gamma x_i)}\left(\delta[I(x_i,\vartheta)]^{\delta-1}-\alpha[I(x_i,\vartheta)]^{\alpha-1}\right)}{(1+\lambda)[I(x_i,\vartheta)]^{\delta}-\lambda[I(x_i,\vartheta)]^{\alpha}}$$

$$- a.(b-1)\sum_{i=1}^{n}\frac{\left\{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}\right\}^{a-1}(x_i)e^{-(\theta x_i{}^{\nu}+\gamma x_i)}}{1-\left\{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}\right\}^{a}}$$

$$\times\left(\delta[I(x_i,\ \vartheta)]^{\delta-1}-\alpha[I(x_i,\ \vartheta)]^{\alpha-1}\right), \tag{27}$$

$$\frac{\partial\tau}{\partial\delta}=\sum_{i=1}^{n}\frac{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta-1}\{\delta\ln[I(x_i,\ \vartheta)]+1\}}{\left[(1+\lambda)\delta[I(x_i,\ \vartheta)]^{\delta-1}-\lambda\alpha[I(x_i,\ \vartheta)]^{\alpha-1}\right]}$$

$$+(a-1)\sum_{i=1}^{n}\frac{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}\ln[I(x_i,\ \vartheta)]}{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}}$$

$$-a.(b-1)\sum_{i=1}^{n}\frac{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}\ln[I(x_i,\ \vartheta)]}{1-\left\{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}\right\}^{a}}, \tag{28}$$

$$\frac{\partial\tau}{\partial\alpha}=\sum_{i=1}^{n}\frac{(-1)\lambda[I(x_i,\ \vartheta)]^{\alpha-1}\{\alpha\ln[I(x_i,\ \vartheta)]+1\}}{\left[(1+\lambda)\delta[I(x_i,\ \vartheta)]^{\delta-1}-\lambda\alpha[I(x_i,\ \vartheta)]^{\alpha-1}\right]}$$

$$-(a-1)\sum_{i=1}^{n}\frac{\lambda[I(x_i,\ \vartheta)]^{\alpha}\,\mathrm{n}[I(x_i,\ \vartheta)]}{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}}$$

$$+a.(b-1)\sum_{i=1}^{n}\frac{\lambda[I(x_i,\ \vartheta)]^{\alpha}\,\mathrm{n}[I(x_i,\ \vartheta)]}{1-\left\{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}\right\}^{a}}, \tag{29}$$

$$\frac{\partial\tau}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\ln\left\{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}\right\}$$

$$-(b-1)\sum_{i=1}^{n}\frac{\left\{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}\right\}^{a}\ln\left((1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}\right)}{1-\left\{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}\right\}^{a}}, \tag{30}$$

and

$$\frac{\partial\tau}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\ln\left\{1-\left\{(1+\lambda)[I(x_i,\ \vartheta)]^{\delta}-\lambda[I(x_i,\ \vartheta)]^{\alpha}\right\}^{a}\right\}. \tag{31}$$

The maximum likelihood estimator $\hat{\underline{\vartheta}}(\hat{\lambda}, \hat{\theta}, \hat{\nu}, \hat{\gamma}, \hat{\delta}, \hat{\alpha}, \hat{a}, \hat{b}) =$ of $\underline{\vartheta} = (\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$ is obtained by solving the nonlinear system of Equations (24) through (31). It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function.

### 7.2. Maximum product spacing estimates

The maximum product spacing (MPS) method has been proposed by Cheng and Amin [9]. This method is based on an idea that the differences (spacing) of the consecutive points should be identically distributed. The geometric mean of the differences is given as

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i}, \tag{32}$$

where the difference $D_i$ is defined as

$$D_i = \int_{x_{(i-1)}}^{x_{(i)}} f(x, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha, a, b)dx; \quad i = 1, 2, \dots, n+1, \tag{33}$$

where $F(x_{(0)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) = 0$ and $F(x_{(n+1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) = 0.$ The MPS estimators $\hat{\lambda}_{PS}$, $\hat{\theta}_{PS}$, $\hat{\nu}_{PS}$, $\hat{\gamma}_{PS}$ $\hat{\delta}_{PS}$, $\hat{\alpha}_{PS}$, $\hat{a}_{PS}$, and $\hat{b}_{PS}$ of $\lambda, \theta, \nu, \gamma, \delta, \alpha, a,$ and $b$ are obtained by maximizing the geometric mean (GM) of the differences. Substituting pdf of KW-NTMW distribution in (33) and taking logarithm of the above expression, we will have

$$\log GM =$$

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F\left(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b\right) - F\left(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b\right)\right]. \tag{34}$$

The MPS estimators $\hat{\lambda}_{PS}$, $\hat{\theta}_{PS}$, $\hat{\nu}_{PS}$, $\hat{\gamma}_{PS}$ $\hat{\delta}_{PS}$, $\hat{\alpha}_{PS}$, $\hat{a}_{PS}$, and $\hat{b}_{PS}$ of $\lambda, \theta, \nu, \gamma, \delta, \alpha, a,$ and $b$ can be obtained as the simultaneous solution of the following nonlinear equations:

$$\frac{\partial \log GM}{\partial \lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}$$

$$\times\left[\frac{F'_\lambda(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F'_\lambda(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \beta, \delta, \alpha, a, b) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}$$

$$\times\left[\frac{F'_\theta(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F'_\theta(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \nu} = \frac{1}{n+1}\sum_{i=1}^{n+1}$$

$$\times\left[\frac{F'_\nu(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F'_\nu(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \gamma} = \frac{1}{n+1}\sum_{i=1}^{n+1}$$

$$\times\left[\frac{F'_\gamma(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F'_\gamma(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \delta} = \frac{1}{n+1}\sum_{i=1}^{n+1}$$

$$\times\left[\frac{F'_\delta(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F'_\delta(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}$$

$$\times\left[\frac{F'_\alpha(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F'_\alpha(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}\right] = 0,$$

$$\frac{\partial \log GM}{\partial a} = \frac{1}{n+1}\sum_{i=1}^{n+1}$$

$$\times\left[\frac{F'_a(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F'_a(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}\right] = 0,$$

and

$$\frac{\partial \log GM}{\partial b} = \frac{1}{n+1}\sum_{i=1}^{n+1}$$

$$\times\left[\frac{F'_b(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F'_b(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - F(x_{(i-1)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}\right] = 0.$$

**7.3. Least square estimates**

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the ordered sample of size $n$ drawn the Kw-NTMW distribution. Then, the expectation of the empirical cumulative distribution function is defined as

$$E[F(X_{(i)})] = \frac{i}{n+1}; \quad i = 1, 2, \ldots, n. \tag{35}$$

The least square estimates $\hat{\lambda}_{LS}, \hat{\theta}_{LS}, \hat{\nu}_{LS}, \hat{\gamma}_{LS}, \hat{\delta}_{LS}, \hat{\alpha}_{LS}, \hat{a}_{LS}$, and $\hat{b}_{LS}$ of $\lambda, \theta, \nu, \gamma, \delta, \alpha, a$, and $b$ are obtained by minimizing

$$Z(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) = \sum_{i=1}^{n}\left[F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - \frac{i}{n+1}\right]^2.$$

Therefore, $\hat{\lambda}_{LS}, \hat{\theta}_{LS}, \hat{\nu}_{LS}, \hat{\gamma}_{LS}, \hat{\delta}_{LS}, \hat{\alpha}_{LS}, \hat{a}_{LS}$, and $\hat{b}_{LS}$ of $\lambda, \theta, \nu, \gamma, \delta, \alpha, a$, and $b$ can be obtained as the solution of the following system of equations:

$$\frac{\partial Z(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{\partial \lambda} = \sum_{i=1}^{n} F'_\lambda(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$$

$$\times\left(F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - \frac{i}{n+1}\right) = 0,$$

$$\frac{\partial Z(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{\partial \theta} = \sum_{i=1}^{n} F'_{\theta}(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$$

$$\times \left( F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial Z(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{\partial \nu} = \sum_{i=1}^{n} F'_{\nu}(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$$

$$\times \left( F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial Z(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{\partial \gamma} = \sum_{i=1}^{n} F'_{\gamma}(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$$

$$\times \left( F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial Z(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{\partial \delta} = \sum_{i=1}^{n} F'_{\delta}(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$$

$$\times \left( F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial Z(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{\partial \alpha} = \sum_{i=1}^{n} F'_{\alpha}(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$$

$$\times \left( F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial Z(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{\partial a} = \sum_{i=1}^{n} F'_{a}(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$$

$$\times \left( F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - \frac{i}{n+1} \right) = 0,$$

and

$$\frac{\partial Z(\lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)}{\partial b} = \sum_{i=1}^{n} F_b'(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b)$$

$$\times \left( F(x_{(i)}, \lambda, \theta, \nu, \gamma, \delta, \alpha, a, b) - \frac{i}{n+1} \right) = 0.$$

These nonlinear can be routinely solved by using Newton's method or fixed point iteration techniques. The subroutines to solve nonlinear optimization problem are available in R. We used nlm( ) package for optimizing (23).

## 8. Applications

In this section, we use real data set to see how the new model works in practice, compare the fits of the Kw-NTMW distribution with others models. In each case, the parameters are estimated by maximum likelihood as described in Section 7, using the R code.

In order to compare the two distribution models, we consider criteria like $-2\mathcal{L}$, AIC (Akaike information criterion), $\text{AIC}_\text{C}$ (corrected Akaike information criterion), and BIC (Bayesian information criterion) for the data set. The better distribution corresponds to smaller $-2\mathcal{L}$, AIC, and $\text{AIC}_\text{C}$ values:

$$\text{AIC} = -2\mathcal{L} + 2k,$$

$$\text{AIC}_\text{C} = -2\mathcal{L} + \left( \frac{2kn}{n-k-1} \right),$$

and

$$\text{BIC} = -2\mathcal{L} + k \log(n),$$

where $\mathcal{L}$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, $k$ is the number of parameters, and $n$ is the sample size.

The data set represents failure time of 50 items reported in Aarset [1].

Some summary statistics for the failure time data are as follows:

| **Min.** | $1^{st}$ **Qu.** | **Median** | **Mean** | $3^{rd}$ **Qu.** | **Max.** |
|---|---|---|---|---|---|
| 0.10 | 13.50 | 48.50 | 45.67 | 81.25 | 86.00 |

**Table 2.** MLEs the measures AIC, $\text{AIC}_\text{C}$, and BICS test to failure time data for the models

| Model | Parameter Estimates | $-\log L$ | AIC | $\text{AIC}_\text{C}$ | BIC |
|---|---|---|---|---|---|
| **KNTAW** | $\lambda = -0.772509$ | 207.6848 | 431.3697 | 434.8818 | 446.6658 |
| | $\theta = 0.00189$ | | | | |
| | $\nu = 0.09235$ | | | | |
| | $\gamma = 0.0152917$ | | | | |
| | $\delta = 1.207579$ | | | | |
| | $\alpha = 0.035739$ | | | | |
| | $a = 0.311286$ | | | | |
| | $b = 0.824264$ | | | | |
| **TEMW** | $\lambda = -0.1640672$ | 236.6535 | 487.6286 | 488.992 | 497.1887 |
| | $\theta = 0.0176781$ | | | | |
| | $\gamma = 0.00193298$ | | | | |
| | $\beta = 0.03926070$ | | | | |
| | $\alpha = 0.949241462$ | | | | |
| **EMW** | $\theta = 0.018673571$ | 238.8143 | 481.307 | 482.1959 | 488.9551 |
| | $\gamma = 0.001822666$ | | | | |
| | $\beta = 0.010505798$ | | | | |
| | $\alpha = 0.703411609$ | | | | |
| **MW** | $\theta = 1.827194$ | 241.0289 | 488.0578 | 488.5795 | 493.7939 |
| | $\gamma = 1.80309$ | | | | |
| | $\beta = 1.000288$ | | | | |

**Table 2.** (Continued)

| | | | | | |
|---|---|---|---|---|---|
| **W** | $\gamma = 0.9489561$ | 240.9796 | 485.959 | 486.2145 | 489.7832 |
| | $\beta = 0.02227559$ | | | | |
| **TEE** | $\hat{\lambda} = 0.84 * 10^{-5}$ | 238.6896 | 483.3793 | 483.9011 | 489.1154 |
| | $\widehat{\beta} = 1.69 * 10^{-4}$ | | | | |
| | $\widehat{\alpha} = 0.30966$ | | | | |
| **EE** | $\widehat{p} = 0.09240$ | 239.9733 | 483.9467 | 484.2021 | 487.7708 |
| | $\widehat{\lambda} = -3.21 * 10^{-5}$ | | | | |
| | $\widehat{\alpha} = 2.23 * 10^{-3}$ | | | | |
| **LE** | $\hat{\lambda} = 0.01364528$ | 280.048836 | 480.0977 | 480.353 | 483.9217 |
| | $\widehat{\beta} = 0.00023990$ | | | | |
| **E** | $\widehat{\beta} = 0.02189828$ | 241.0677 | 484.1354 | 484.2187 | 486.0474 |

These results indicate that the Kw-NTMW model has the lowest AIC and $AIC_C$ and BIC values among the fitted models. The values of these statistics indicate that the Kw-NTMW model provides the best fit to this data.

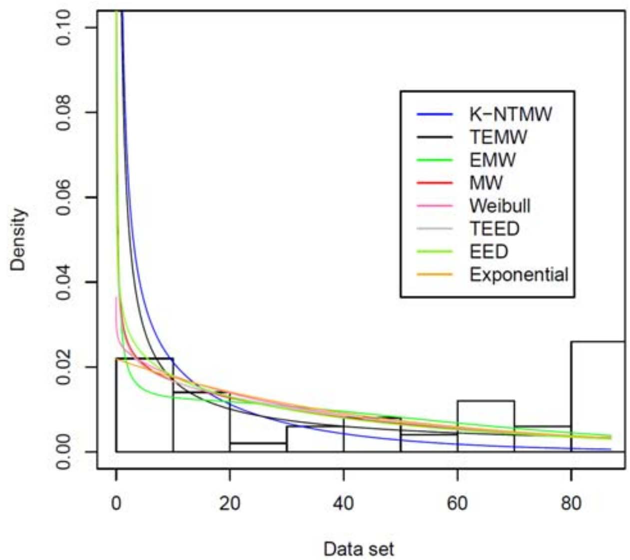


**Figure 5.** Estimated densities of the data set.

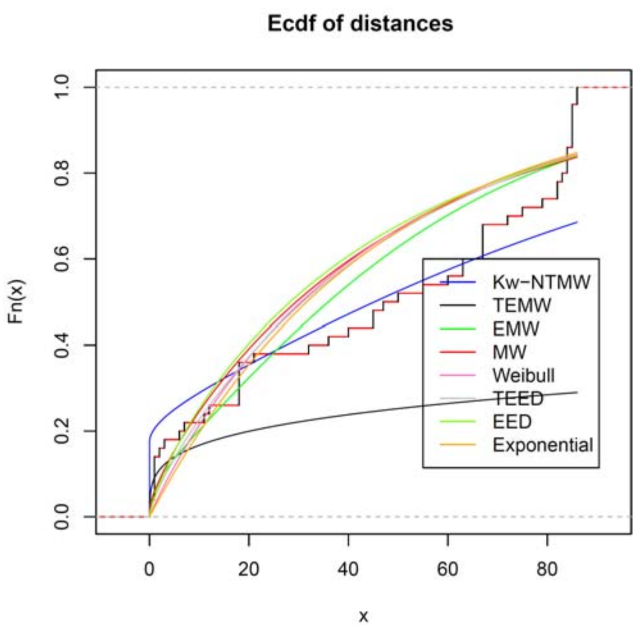


**Figure 6.** Empirical, fitted, Kw-NTMW, TEMW, EMW, MW, Weibull, TEE, EE, and exponential distributions of the data set.

## 9. Concluding Remarks

In the present study, we have introduced a new generalization of the modified Weibull distribution, called the Kumaraswamy new transmuted modified Weibull distribution. We refer to the new model as the Kw-NTMW distribution and study some of its mathematical and statistical properties. We provide the pdf, the cdf, and the hazard rate function of the new model, explicit expressions for the moments. The model parameters are estimated by maximum likelihood. The new model is compared with some models and provides consistently better fit than other classical lifetime models. We hope that the proposed model will serve as a reference and help to advance future research in this area.

## References

[1] M. V. Aarset, How to identify a bathtub hazard rate, IEEE Transactions on Reliability 36(1) (1987), 106-108.

[2] A. Al-Babtain, A. A. Fattah, A. N. Ahmed and F. Merovci, The Kumaraswamy-transmuted exponentiated modified Weibull distribution, Accepted for Communications in Statistics-Theory and Methods (2015).

[3] M. S. Ammar and Z. Mazen, Modified Weibull distribution, Applied Sciences 11 (2009), 123-136.

[4] G. R. Aryal and C. P. Tsokos, Transmuted Weibull distribution: A generalization of the Weibull probability distribution, European Journal of Pure and Applied Mathematics 4(2) (2011), 89-102.

[5] S. Ashour and M. Eltehiwy, Transmuted exponentiated modified Weibull distribution, International Journal of Basic and Applied Sciences 2(3) (2013), 258-269.

[6] R. E. Barlow and F. Proschan, Statistical Theory or Reliability and Life Testing: Probability Models, Tobe with, 1981.

[7] H. W. Block and T. H. Savits, Burn-in, Statistical Science 12(1) (1997), 1-19.

[8] D. S. Chang, Optimal burn-in decision for products with a unimodal failure rate function, European Journal of Operational Research 126(3) (2000), 534-540.

[9] R. C. H. Cheng and N. A. K. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, Journal of the Royal Statistical Society, Series B (Methodological) (1983), 394-403.

[10] G. M. Cordeiro and M. de Castro, A new family of generalized distributions, Journal of Statistical Computation and Simulation 81(7) (2011), 883-898.

[11] G. M. Cordeiro, E. M. Ortega and S. Nadarajah, The Kumaraswamy Weibull distribution with application to failure data, Journal of the Franklin Institute 347(8) (2010), 1399-1429.

[12] G. M. Cordeiro, E. M. Ortega and G. O. Silva, The Kumaraswamy modified Weibull distribution: Theory and applications, Journal of Statistical Computation and Simulation 84(7) (2014), 1387-1411.

[13] G. M. Cordeiro, R. R. Pescim and E. M. Ortega, The Kumaraswamy generalized half-normal distribution for skewed positive data, J. Data Sci. 10 (2012), 195-224.

[14] I. Elbatal, Exponentiated modified Weibull distribution, Economic Quality Control 26(2) (2011), 189-200.

[15] I. Elbatal, Kumaraswamy generalized linear failure rate distribution, Indian Journal of Computational & Applied Mathematics 1(1) (2013), 61-78.

[16] A. E. Gomes, C. Q. de-Silva, G. M. Cordeiro and E. M. Ortega, A new lifetime model: The Kumaraswamy generalized Rayleigh distribution, Journal of Statistical Computation and Simulation 84(2) (2014), 290-309.

[17] M. Greenwich, A unimodal hazard rate function and its failure distribution, Statistical Papers 33(1) (1992), 187-202.

[18] R. C. Gupta, P. L. Gupta and R. D. Gupta, Modeling failure time data by Lehman alternatives, Communications in Statistics-Theory and Methods 27(4) (1998), 887-904.

[19] R. D. Gupta and D. Kundu, Generalized exponential distributions, Australian & New Zealand Journal of Statistics 41(2) (1999), 173-188.

[20] R. D. Gupta and D. Kundu, Exponentiated exponential family: An alternative to gamma and Weibull distributions, Biometrical Journal 43(1) (2001), 117-130.

[21] R. Jiang, P. Ji and X. Xiao, Ageing property of unimodal failure rate models, Reliability Engineering & System Safety 79(1) (2003), 113-116.

[22] M. C. Jones, Kumaraswamy's distribution: A beta-type distribution with some tractability advantages, Statistical Methodology 6(1) (2009), 70-81.

[23] W. D. Kelton and A. M. Law, Simulation Modeling and Analysis, Boston, MA: McGraw Hill, 2000.

[24] M. S. Khan and R. King, Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution, European Journal of Pure and Applied Mathematics 6(1) (2013), 66-88.

[25] P. Kumaraswamy, A generalized probability density function for double-bounded random processes, Journal of Hydrology 46(1) (1980), 79-88.

[26] D. Kundu and M. Z. Raqab, Generalized Rayleigh distribution: Different methods of estimations, Computational Statistics & Data Analysis 49(1) (2005), 187-200.

[27] C. D. Lai, M. Xie and D. N. P. Murthy, A modified Weibull distribution, IEEE Transactions on Reliability 52(1) (2003), 33-37.

[28] D. V. Lindley, Fiducial distributions and Bayes' theorem, J. Royal Stat. Soc. Series B 20 (1958), 102-107.

[29] M. M. Mansour, M. A. Enayat, S. M. Hamed and M. S. Mohamed, A new transmuted additive Weibull distribution based on a new method for adding a parameter to a family of distributions, Accepted for JAMS (2015).

[30] F. Merovci, Transmuted exponentiated exponential distribution, Mathematical Sciences and Applications E-Notes 1(2) (2013), 112-122.

[31] F. Merovci, Transmuted Rayleigh distribution, Austrian Journal of Statistics 42(1) (2013), 21-31.

[32] F. Merovci, Transmuted generalized Rayleigh distribution, Journal of Statistics Applications and Probability 3(1) (2014), 9-20.

[33] G. S. Mudholkar and D. K. Srivastava, The exponentiated Weibull family for analyzing bathtub failure-rate data, IEEE Transactions on Reliability 42(2) (1993), 299-302.

[34] G. S. Mudholkar, D. K. Srivastava and M. Freimer, The exponentiated Weibull family: A reanalysis of the bus-motor-failure data, Technometrics 37(4) (1995), 436-445.

[35] D. P. Murthy, M. Xie and R. Jiang, Weibull Models, Vol. 505, John Wiley & Sons, 2004.

[36] H. Pham and C. D. Lai, On recent generalizations of the Weibull distribution, IEEE Transactions on Reliability 56(3) (2007), 454-458.

[37] A. M. Sarhan and D. Kundu, Generalized linear failure rate distribution, Communications in Statistics-Theory and Methods 38(5) (2009), 642-660.

[38] A. M. Sarhan and M. Zaindin, Modified Weibull distribution, Applied Sciences 11(1) (2009), 123-136.

[39] E. W. Stacy, A generalization of the gamma distribution, The Annals of Mathematical Statistics (1962), 1187-1192.

[40] R. C. Team, R: A Language and Environment for Statistical Computing, R Foundation for Statistical Computing, Vienna, Austria, 2012. ISBN 3-900051-07-0.

[41] W. Weibull, A statistical theory of the strength of materials, IVA Handlingar (Royal Swedish Academy of Engineering Sciences, Proceedings) nr, 151 (1939).

[42] M. Xie and C. D. Lai, Stochastic Ageing and Dependence for Reliability, Springer, New York, 2006.

[43] M. Zaindin and A. M. Sarhan, Parameters estimation of the modified Weibull distribution, Applied Mathematical Sciences 3(11) (2009), 541-550.

[44] T. Zhang, M. Xie, L. C. Tang and S. H. Ng, Reliability and modeling of systems integrated with firmware and hardware, International Journal of Reliability, Quality and Safety Engineering 12(03) (2005), 227-239.

■